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## ABSTRACT

A single server queueing system with Poisson arrival at rate $\lambda$ and a finite queue capacity $N$ is considered. The service can be performed using one of $K$ available service rates $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_K$. When service rate $\mu_k$ is in effect the service time is a random variable exponentially distributed with expected service time $1/\mu_k$, and a cost rate $c(\mu_k)$ is incurred. For each customer served, a reward of $A$ will be earned. The problem is to choose service rate in order to minimize the expected discounted or expected long run average cost. We first formulate the problem as a semi-Markov decision process with the state as the number of customers in the system. It is then shown that the optimal policy is switchover or reverse switchover depending upon whether or not the system is profitable. A switchover policy is a policy which uses higher rates in the higher states and a reverse switchover policy uses lower rates in the higher states.

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# CHAPTER I

# INTRODUCTION

## 1.1 Description of Problem

Optimization problems in queueing have received a large amount of attention during the past few years. Up to a decade ago most of the works in queueing were concerned about the descriptive aspect of the problem, namely deriving mathematical characteristics and structure of the system such as stationary distributions, waiting time distribution, expected number of people in the system, etc. In almost all branches of industries where queueing plays a role, in addition to the descriptive models, the manager is confronted with the optimization of design and control of the system. With the introduction of Markovian decision processes and its relation with queueing the optimization of queueing system gained a new momentum.

In a general optimization problem, choosing a policy or value of a variable in order to minimize a function is sought. In queueing these variables can enter in the arrival stream or service facility. The latter is encountered more in practice and therefore, more attention is given to these problems, but a number of works can be found which controlling the arrival process is the target of the optimization. In some case even controlling both arrival and service variables have been the target.

In this thesis we are concerned with the second type of problem. Namely, the decisions are made on the service facility.

We are considering a queueing system with a single servicing facility and a limited space $N$ for waiting customers. Those customers who, upon arrival, find the system with $N$ persons do not stop. The arrival stream is a Poisson process with rate $\lambda$. The service can be performed with one of the $K$ existing service rates $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_K$. When service with rate $\mu_k$ is used the service time is a random variable which is exponentially distributed with expected service time $1/\mu_k$ . The server can switch a service rate to another at the time of an arrival or at the time of a departure. These switches, if made, must optimize the overall expected discounted or average cost for the given cost structure. The cost structure considered is as follows.

(a) A service cost rate $c(\mu_k)$ , the cost of servicing at rate $\mu_k$ . It is assumed that $c(\mu_k)$ is nondecreasing in $\mu_k$

$$c(\mu_1) \leq c(\mu_2) \leq \ldots \leq c(\mu_K) .$$

(b) The server will receive a reward $A$ for each customer he serves.

With this cost structure the optimal policy is found to have a simple form. If the state of the system is defined as the number of people in the system, then, e.g., in the case of two rates $\mu_1 < \mu_2$ , it will be shown that the optimal policy prescribes $\mu_1$ up to a state $J$ , $0 \leq J \leq N$ , and $\mu_2$ for all states higher than $J$ , which we will call a switchover policy. The case of $K \geq 2$ service rates is also analyzed and a simple policy is found.

With the same cost structure the case of "hesitating customers" is also analyzed, where the arriving customer will stop with a probability $p_i$ if there are $i$ person in the system. It is shown that for $p_i$ monotone the same type of policy is optimal.

Although complete results are not found for the expected discounted model, Chapter II is devoted to this model since some of the results in this case are needed in the average cost case.

## 1.2 Review of Literature

As was mentioned in the previous section the problem of optimization in queueing has been given a considerable amount of attention during recent years.

In the earliest papers assumptions were made about the class of feasible policies and within this restricted class the best policies were found. For example in Yadin and Naor [18], they find the optimal queue size at which to turn the server on assuming that the policy is to turn the server on when queue length reaches a certain level.

Heyman [ 3 ] in his work considered the model of Yadin and Naor and attempted to find the form of an optimal policy in an M/G/1 queue without making a priori assumptions about the policy.

Thatcher [17] consideres an M/G/1 queue with two service rates. He does not assume a queue limit but does assume a holding cost which is dependent upon the number of customers in the system. He first finds the best switch-over policy and then he proves its optimality among all policies.

In general when the form of optimal policy is the target,the theory of stochastic decision processes plays an important role.

As was mentioned before a decision might be made on arrival, service or both. Work of Miller [10], Lippman and Ross [ 8 ] are of the first catagory. In Miller the decision is made to reject or to accept an arrival which can be from one of the $n$ possible types, where the nth type customer offers a reward $R_n$ . The queueing system he considers is M/M/C, where no queue is allowed. Lippman and Ross generalize Miller's problem to a general arrival, service, and reward distributions with an infinite number of arrival types but only one server. In both cases simple policies are found in order to maximize the expected average return.

Work of Crabill [2], McGill [ 9 ] and Mitchell [11] are of the second catagory where the decisions are made on servicing facility.

Crabill found the optimal servicing rates for an infinite queue. The system he considers is M/M/1 and the policy he finds is switchover.

McGill considers a multi-channel queue with changing number of servers as a function of persons in the system.

Mitchell, in the work still in progress, considers a single server queue with Poisson arrivals, where each customer in the system requires a random length of service which becomes known at the time when customer enters service. The decision is made as to which of two rates to use.

Klimov in [ 7 ] considers an m servers queue with nonstationary Poisson arrivals and nonstationary exponential service. One of the cases considered there is where the arrival and service rate are not known but must be found in order to optimize an objective function. The method used is optimal control.

## 1.3 Notations and Definitions, Mathematical Background

In the next chapters we will formulate our queueing problem as a semi-Markov decision process. The early contributers of these processes are Jewell [ 6 ] and Howard [ 4 ], [ 5 ] and recently Ross [12], [13]. The notations and definitions used here are of Ross.

Definition:

A *semi-Markov decision process* is a process which is observed in each review point and is found to be in one of a possible number of states. The set of all possible states is called the *state space* $S$ . After observing the state an action, $a$ , must be made from a set of possible actions. The set of all possible actions is called the *action space* $A$ . Whenever the state is $i$ and action $a$ is chosen, then

(i) A transition to state $j$ occurs with the probability $P_{ij}(a)$ .

(ii) If the next state of the system is $j$ , then the time until the transition from $i$ to $j$ is a random variable with distribution $F_{ij}(a)$ .

(iii) The action $a$ taken in state $i$ will specify the cost incurred in the next stage of the process.

A *policy* is a rule for choosing actions when the current state and the past history of the process is given.

A *stationary policy* is a nonrandomized policy where the action chosen at a time only depends on the state of the process at that time.

Average Cost Results

Let $Z(t)$ be the total cost incurred by time $t$ and $Z_n$ be the cost incurred during the nth transition interval and $\tau_n$ be the length of this interval. For each policy $\pi$ and state $i$, we define

$$\phi_\pi^1(i) = \lim_{t\to\infty} E_\pi\left[\frac{Z(t)}{t} / X_1 = i\right] \tag{1.1}$$

and

$$\phi_\pi^2(i) = \lim_{n\to\infty} \frac{E_\pi\left[\sum_{j=1}^{n} Z_j / X_1 = i\right]}{E_\pi\left[\sum_{j=1}^{n} \tau_j / X_1 = i\right]} \tag{1.2}$$

where, $X_1$ is the initial state of the system. Although, $\phi^1$ represents the long-run average expected cost, it turns out that $\phi^2$ is a "nicer" criteria, in analytical sense, to work with. It also turns out that under some reasonable conditions these two criteria are equal.

Let $T$ be the time of the first return to state $i$, and $f$ any stationary policy and suppose that

$$E_f[T/X_1 = i] < \infty \, . \tag{1.3}$$

Then it can be shown (see Ross [12]) that

$$\phi_f^1(i) = \phi_f^2(i) \, .$$

A method to find an optimal stationary policy using $\phi^2$ is given in [12]. Thus, if (1.3) is satisfied then, this same policy is optimal for $\phi^1$.

Now, let $\bar{c}(i,a)$ denote the expected cost incurred in a transition interval that begins with action $a$ being taken in state $i$. Also, let $\bar{\tau}(i,a)$ denote the expected length of such a transition interval. It should be noted that, when the relevant criteria is given by Equation (1.2), then without loss of generality we may assume that the cost incurred in such an interval (and the length of such an interval) is with probability one $\bar{c}(i,a)(\bar{\tau}(i,a))$. This will be assumed here on.

Note that this implies that $V_\alpha(i)$, the minimal total expected $\alpha$-discounted cost starting from state $i$, satifies

$$(1.4) \qquad V_\alpha(i) = \min_a \left\{ \bar{c}(i,a) + e^{-\alpha\bar{\tau}(i,a)} \sum_j P_{ij}(a) V_\alpha(j) \right\}, \qquad i \geq 0 .$$

The following theorem may be found in Ross [13].

<u>Theorem</u>:

If $\bar{c}(i,a)$ is bounded, and if there exists an $N < \infty$ such that

$$|V_\alpha(i) - V_\alpha(0)| < N \qquad \text{for all } a \text{, all } i$$

then, there exists a bounded function $h(i)$ and a constant $g$ such that

$$(1.5) \qquad h(i) = \min_a \left\{ \bar{c}(i,a) + \sum_{j=0}^{\infty} P_{ij}(a) h(j) - g\bar{\tau}(i,a) \right\} \qquad i \geq 0 .$$

and if $\pi^*$ is a policy which, for each $i$, prescribes an action which minimizes the right side of (1.5) then

$$g = \phi^2_{\pi^*}(i) = \min_\pi \phi^2_\pi(i) \qquad \text{all } i \geq 0 .$$

This theorem will be used in later chapters.

# CHAPTER II

# DISCOUNTED MODEL

## 2.1 Introduction

In this chapter we will formulate the queueing system as a semi-Markov decision process. We assume a queueing system with Poisson arrivals at rate $\lambda$ , and a single exponential service facility. The service facility can perform its duty using one of the $K$ available service rates, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_K$ . When service rate $\mu_k$ is in effect, the service time is a random variable which is exponentially distributed with expected service time $1/\mu_k$ . Those arrivals finding system with $N$ persons will not stop and will be lost. In other words the queueing facility can only hold $N$ persons at a time. The problem will be formulated in two cases. First, the operator can only switch at the time of departure of a customer in the system. Second, where the operator can switch at either an arrival or at a departure time of a customer. These switches, if made, must minimize the overall expected $\alpha$-discounted cost of the system. The following cost structure is considered:

(i) a service cost rate $c(\mu_k)$ , the cost rate incurred when service rate $\mu_k$ is in effect. It is assumed that $c(\mu_k)$ is nondecreasing in $\mu_k$ .

(ii) For each customer served a reward of $A$ will be earned.

In Section 2 the problem will be formulated as a semi-Markov decision process and in Section 3 some properties of optimal value function defined in Section 2 will be studied. In Section 4 some special cases will be analyzed and a sufficient condition for the existance of a simple switchover policy will be given.

## 2.2 Semi-Markov Decision for Queueing

We must define the state of the system, the available actions, and the probabilistic law or the law of otion. The state of the system is defined as the number of customers in the queue. The state space $S$, is thus a set of $N + 1$ nonnegative integers $S = \{0,1, \ldots, N\}$. The action space $K$ is defined as a set of $K$ nonnegative integers $K = \{1,2, \ldots, K\}$, where action $k$ corresponds to rate $\mu_k$.

*The transition probabilities*, if switching is only allowed at the departure points can be easily found to be as follows:

$$(2.1)\quad P_{ij}(a) = \begin{cases} 0 & j < i-1 \quad \text{all } i \geq 1 \\ \dfrac{\lambda^t \mu_a}{(\lambda + \mu_a)^{t+1}} \quad t = 0,1, \ldots, N-i-1 & j = i+t-1 \quad i \geq 1 \\ \displaystyle\sum_{t=N-i}^{\infty} \frac{\lambda^t \mu_a}{(\lambda + \mu_a)^{t+1}} & j = N-1 \quad i \geq 1 \\ 0 & j \geq N \quad i \geq 1 \\ 1 & j = 1 \quad i = 0 . \end{cases}$$

If switching is allowed in both departure and arrival points then

$$P_{ij}(a) = \begin{cases} 0 & j \neq i-1, i+1 \quad i \geq 1 \\ \dfrac{\mu_a}{\lambda + \mu_a} & j = i-1 \quad i \geq 1 \\ \dfrac{\lambda}{\lambda + \mu_a} & j = i+1 \quad i \geq 1 \\ 1 & j = 1 \quad i = 0 . \end{cases}$$

When the first set of transition probabilities are in effect then we will say that we are in Case 1, and similarly for Case 2. Now for Case 1, if

$V_\alpha(i)$ is the optimal value function as defined in Section 3 of the previous chapter, then

(2.3)

$$V_\alpha(i) = \min_a \left\{ \frac{c(\mu_a)}{\mu_a} - A + e^{-\frac{\alpha}{\mu_a}} \sum_{j=0}^{N-i-1} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} V_\alpha(i + j - 1) + e^{-\frac{\alpha}{\mu_a}} V_\alpha(N - 1) \sum_{j=N-i}^{\infty} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} \right\} \quad i = 1, \ldots, N-1$$

$$V_\alpha(N) = \min_a \left\{ \frac{c(\mu_a)}{\mu_a} - A + e^{-\frac{\alpha}{\mu_a}} V_\alpha(N - 1) \right\}$$

$$V_\alpha(0) = e^{-\frac{\alpha}{\lambda}} V_\alpha(1)$$

where, $\frac{c(\mu_a)}{\mu_a} - A$ is the cost incurred in a transition interval of length $1/\mu_a$ , the expected transition time when rate $\mu_a$ is in effect. In other words, $V_\alpha(i)$ is the minimal expected $\alpha$-discounted cost starting from state $i$ , where the operator can only switch at departure points. In the case where switching is allowed at any time (that is, in Case 2) the optimal value function $\bar{V}_\alpha(i)$ satisfies

(2.4)

$$\bar{V}_\alpha(i) = \min_a \left\{ \frac{c(\mu_a) - \mu_a A}{\lambda + \mu_a} + e^{-\frac{\alpha}{\lambda + \mu_a}} \frac{\lambda}{\lambda + \mu_a} \bar{V}_\alpha(i + 1) + e^{-\frac{\alpha}{\lambda + \mu_a}} \frac{\mu_a}{\lambda + \mu_a} \bar{V}_\alpha(i - 1) \right\} \quad i = 1, \ldots, N-1$$

$$\bar{V}_\alpha(N) = \min_a \left\{ \frac{c(\mu_a)}{\mu_a} - A + e^{-\frac{\alpha}{\mu_a}} \bar{V}_\alpha(N - 1) \right\} .$$

$$\bar{V}_\alpha(0) = e^{-\frac{\alpha}{\lambda}} V(1) .$$

Of course, $\bar{V}_\alpha(i)$ and $V_\alpha(i)$ are different.

## 2.3 Properties of the Optimal Value Function

In this section some properties of the optimal value function $V_\alpha(i)$ will be studied.

Lemma 1:

$V_\alpha(i) \geq 0$ for all $i$, if and only if $\frac{c(\mu_a)}{\mu_a} \geq A$ for all $a$.

Proof:

The proof is by induction, define

$$V_\alpha(i,1) = \min_a \left\{ \frac{c(\mu_a)}{\mu_a} - A \right\} \tag{2.5}$$

and recursively

$$\begin{aligned} V_\alpha(i,n) = \min_a \Bigg\{ & \frac{c(\mu_a)}{\mu_a} - A + \sum_{j=0}^{N-i-1} e^{-\frac{\alpha}{\mu_a}} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} V_\alpha(i + j - 1, n - 1) \\ & + e^{-\frac{\alpha}{\mu_a}} V_\alpha(N - 1, n - 1) \sum_{j=N-i}^{\infty} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} \Bigg\} . \end{aligned} \tag{2.6}$$

Functions $V_\alpha(i,n)$ will converge to $V_\alpha(i)$ as $n$ goes to infinity, see [13].

(i) If $\frac{c(\mu_a)}{\mu_a} \geq A$ for all $a$, then clearly $V_\alpha(i,1) \geq 0$.

Now assume $V_\alpha(i,m) \geq 0$ for all $m \leq n - 1$. Then, by induction assumption all terms in the right side of (2.6) are nonnegative. Hence, $V_\alpha(i,n) \geq 0$ for all $n$, and in the limit $V_\alpha(i) \geq 0$. ∎

(ii) If $V_\alpha(i) \geq 0$, for all $i$ then $\frac{c(\mu_a)}{\mu_a} \geq A$ for all $a$.

Assume the contrary, that $\frac{c(\mu_k)}{\mu_k} < A$ for some $k$, then $V(i,1) + \varepsilon < 0$ for some $\varepsilon > 0$, and using this rate $\mu_k$ in all states implies $V_\alpha(i,n) < -\varepsilon$ for all $i$ and $n$, or in the limit $V_\alpha(i) < 0$ which contradicts the assumption.

Q.E.D.

An immediate conclusion of this lemma is if $\frac{c(\mu_k)}{\mu_k} < A$ for some $k$ then $V_\alpha(i) < 0$ for all $i$.

<u>Lemma 2</u>:

$V_\alpha(i)$ is nonincreasing if and only if $\frac{c(\mu_a)}{\mu_a} < A$ for some $a$.

<u>Proof</u>:

We prove this lemma by induction as of the previous lemma.

(i) If $\frac{c(\mu_a)}{\mu_a} < A$ then $V_\alpha(i)$ is nonincreasing.

From Equation (2.5)

$$V_\alpha(i,1) = V_\alpha(i + 1,1) \qquad \text{for all } i \geq 1$$

and

$$V_\alpha(0,1) = e^{-\alpha\left(\frac{1}{\lambda} + \frac{1}{\mu_a^*}\right)} V(1,1)$$

where, $\mu_a^*$ is the rate which minimizes $\left\{\frac{c(\mu_a)}{\mu_a} - A\right\}$. By assumption $V(1,1) < 0$. Hence, $V_\alpha(0,1) > V(1,1) = V(i,1)$ $i > 1$ and the lemma is true for $n = 1$. Now assume the lemma is true for all $m \leq n - 1$, then

$$V_\alpha(i,n) = \min_a \left\{ \frac{c(\mu_a)}{\mu_a} - A + e^{-\frac{\alpha}{\mu_a}} \sum_{j=0}^{N-i-1} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} V_\alpha(i + j - 1, n - 1) \right.$$

(2.7)

$$\left. + e^{-\frac{\alpha}{\mu_a}} V_\alpha(N - 1, n - 1) \sum_{j=N-i}^{\infty} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} \right\}$$

$$V_\alpha(i + 1,n) = \min_a \left\{ \frac{c(\mu_a)}{\mu_a} - A + e^{-\frac{\alpha}{\mu_a}} \sum_{j=0}^{N-i-2} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} V_\alpha(i + j, n - 1) \right.$$

(2.8)

$$\left. + e^{-\frac{\alpha}{\mu_a}} V(N - 1, n - 1) \sum_{j=N-i-1}^{\infty} \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} \right\} .$$

For any fixed $a$, the right side of (2.7) is larger than the right side of (2.8) by induction assumption. Hence,

$$V_\alpha(i,n) \geq V_\alpha(i + 1,n) \qquad \text{for all } n \text{ and } i$$

and in the limit $V_\alpha(i) \geq V_\alpha(i + 1)$ .

(ii) If $V_\alpha(i) < V_\alpha(i + 1)$ then $\frac{c(\mu_a)}{\mu_a} \geq A$ for all $a$ .

If not let $k$ be an action such that $\frac{c(\mu_k)}{\mu_k} < A$ . Then $V(i,n) < 0$ , and by part (i) of the proof $V_\alpha(i + 1) \leq V_\alpha(i)$ is a contradiction.

<u>Lemma 3</u>:

$V_\alpha(i)$ is nondecreasing if and only if $\frac{c(\mu_a)}{\mu_a} \geq A$ for all $a$ .

<u>Proof</u>:

Proof of this lemma is similar to the previous one, and is omitted here.

Remark:

Lemmas 1, 2, and 3 can be proved for functions $\bar{V}_\alpha(i)$ . The proof is similar to those lemmas and is omitted here.

The following lemma will be useful in the next chapter.

Lemma 4:

(a) There exists an $M < \infty$ such that

$$|V_\alpha(i) - V_\alpha(0)| < M \qquad \text{for all } \alpha \text{ , and all } i \text{ .}$$

(b) There exists an $\bar{M} < \infty$ such that

$$|\bar{V}_\alpha(i) - \bar{V}_\alpha(0)| < \bar{M} \qquad \text{for all } \alpha \text{ , and all } i \text{ .}$$

Proof:

(a) Let $(r_1, \ldots, r_N)$ be the optimal service rates used in states $(1, \ldots, N)$ then the embeded Markov chain corresponding to departure points is irreducible and positive recurrent, see (2.1). Let $K_i$ be the expected number of transition starting from state $i$ before reaching state $0$ , then $K_i < \infty$ for all $i$ . Let $K = \max_i K_i$ , for the case $V_\alpha(i) \leq 0$

$$V_\alpha(i) \geq K \min_a \left\{\frac{c(\mu_a)}{\mu_a} - A\right\} + V_\alpha(0)$$

$$V_\alpha(i) - V_\alpha(0) \geq K \min_a \left\{\frac{c(\mu_a)}{\mu_a} - A\right\} .$$

The left side of the above inequality is also bounded by $0$ . Hence

$$|V_\alpha(i) - V_\alpha(0)| \leq K \left|\min_a \left\{\frac{c(\mu_a)}{\mu_a} - A\right\}\right|$$

for the case $V_\alpha(i) \geq 0$

$$V_\alpha(i) - V_\alpha(0) \leq K \max_a \left\{\frac{c(\mu_a)}{\mu_a} - A\right\} .$$

or

$$|V_\alpha(i) - V_\alpha(0)| \leq K \max_a \left\{\frac{c(\mu_a)}{\mu_a} - A\right\} .$$

Now let

$$M = \max \left\{K \left|\min_a \left(\frac{c(\mu_a)}{\mu_a} - A\right)\right| ; K \max_a \left\{\frac{c(\mu_a)}{\mu_a} - A\right\}\right\}$$

then

$$|V_\alpha(i) - V_\alpha(0)| < M . \tag{2.9}$$

(b) The proof of this part is similar to the first part using the transition probabilities (2.2), and arguing in the same way.

Q.E.D.

## 2.4 Some Properties of Optimal Policy

In this section some special cases will be analyzed and a conditional theorem for the existance of a switchover policy will be given. We define $f^*(i)$ to be the optimal action at state $i$. The following lemmas and theorems are proved for Case 1.

Lemma 5:

If $f^*(N) = k$ and $A > \frac{c(\mu_a)}{\mu_a}$ for some $a$, then $f^*(i) \leq k$ for all $i \leq N$, in other words the largest rate is used in the largest state.

<u>Proof</u>:

By assumption

$$\frac{c(\mu_k)}{\mu_k} - A + e^{-\frac{\alpha}{\mu_k}} V_\alpha(N-1) \le \frac{c(\mu_a)}{\mu_a} - A + e^{-\frac{\alpha}{\mu_a}} V_\alpha(N-1) \qquad \text{all } a \ne k .$$

Let $g(i,a,k)$ be the difference between two expected future costs when action $a$ or $k$ is taken in $i$, then

$$(2.10) \qquad g(N,a,k) = \frac{c(\mu_a)}{\mu_a} - \frac{c(\mu_k)}{\mu_k} + \left(e^{-\frac{\alpha}{\mu_a}} - e^{-\frac{\alpha}{\mu_k}}\right) V(N-1) .$$

By assumption $g(N,a,k) \ge 0$ for all $a \ne k$. Assume in the contrary that there exists state $i$ such that $f^*(i) = a > k$ then

$$g(i,a,k) = \frac{c(\mu_a)}{\mu_a} - \frac{c(\mu_k)}{\mu_k} + \sum_{j=0}^{N-i-1} \left[\frac{\lambda^j \mu_a}{(\lambda+\mu_a)^{j+1}} e^{-\frac{\alpha}{\mu_a}} - \frac{\lambda^j \mu_k}{(\lambda+\mu_k)^{j+1}} e^{-\frac{\alpha}{\mu_k}}\right],$$

$$(2.12) \qquad V_\alpha(i+j-1) + V_\alpha(N-1) \sum_{j=N-i}^{\infty} \left[\frac{\lambda^j \mu_a}{(\lambda+\mu_a)^{j+1}} e^{-\frac{\alpha}{\mu_a}} - \frac{\lambda^j \mu_k}{(\lambda+\mu_k)^{j+1}} e^{-\frac{\alpha}{\mu_k}}\right]$$

where $g(i,a,k) < 0$. In (2.10) substitute for $e^{-\frac{\alpha}{\mu_a}}$ and $e^{-\frac{\alpha}{\mu_k}}$ the following:

$$e^{-\frac{\alpha}{\mu_a}} = e^{-\frac{\alpha}{\mu_a}} \sum_{j=0}^{\infty} \frac{\lambda^j \mu_a}{(\lambda+\mu_a)^{j+1}} , \quad e^{-\frac{\alpha}{\mu_k}} = e^{-\frac{\alpha}{\mu_k}} \sum_{j=0}^{\infty} \frac{\lambda^j \mu_k}{(\lambda+\mu_k)^{j+1}}$$

then

$$g(i,a,k) - g(N,a,k) = \sum_{j=0}^{N-i-1} \left[ \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} e^{-\frac{\alpha}{\mu_a}} - \frac{\lambda^j \mu_k}{(\lambda + \mu_k)^{j+1}} e^{-\frac{\alpha}{\mu_k}} \right] \times \tag{2.13}$$
$$[V_\alpha(i + j - 1) - V_\alpha(N - 1)] .$$

For nonincreasing $V_\alpha(i)$ , $V_\alpha(i + j - 1) - V_\alpha(N - 1)$ is nonincreasing and nonnegative. And the function $h(j)$ defined as

$$h(j) = \frac{\lambda^j \mu_a}{(\lambda + \mu_a)^{j+1}} e^{-\frac{\alpha}{\mu_a}} - \frac{\lambda^j \mu_k}{(\lambda + \mu_k)^{j+1}} e^{-\frac{\alpha}{\mu_k}} \qquad a > k$$

changes sign from positive to negative (note that, function $S(x) = \frac{\lambda^j x}{(\lambda + x)^{j+1}} e^{-\frac{\alpha}{x}}$ is increasing for $j \leq J$ , some $J$ , $0 < J \leq N$ , and decreasing for $j > J$) and since

$$\sum_{j=0}^{\infty} h(j) = e^{-\frac{\alpha}{\mu_a}} - e^{-\frac{\alpha}{\mu_k}} > 0$$

then the positive terms in (2.13) dominate the negative terms. Hence,

$$g(i,a,k) - g(N,a,k) \geq 0$$

$$g(i,a,k) \geq g(N,a,k) \geq 0$$

which contradicts the assumption that $f^*(i) = a$ .

Q.E.D.

Lemma 6:

Let there exist an $I$ such that the action space $K = \{1,2, \ldots, K\}$ can be partitioned into two subsets $K_1 = \{1,2, \ldots, I\}$ and

$K_2 = \{I+1, \ldots, K\}$ where, for all $a \in K_1$ , $\frac{c(\mu_a)}{\mu_a} < A$ and for all $a \in K_2$ , $\frac{c(\mu_a)}{\mu_a} \geq A$ . Then, the optimal policy does not include any action from $K_2$ .

Proof:

From Lemma 5 if $f^*(N) = a$ and $a \in K_1$ then optimal policy does not include elements of $K_2$ . Let us assume $f^*(N) = a$ and $a \in K_2$ then

$$V_\alpha(N) = \frac{c(\mu_a)}{\mu_a} - A + e^{-\frac{\alpha}{\mu_a}} V_\alpha(N-1) \tag{2.14}$$

substract $V_\alpha(N-1)$ from both sides of (2.14)

$$V_\alpha(N) - V_\alpha(N-1) = \frac{c(\mu_a)}{\mu_a} - A + \left(e^{-\frac{\alpha}{\mu_a}} - 1\right) V(N-1) \ . \tag{2.15}$$

By Lemma 2 $V_\alpha(N) - V_\alpha(N-1) < 0$ and by assumption $\frac{c(\mu_a)}{\mu_a} - A \geq 0$ , since $e^{-\frac{\alpha}{\mu_a}} - 1 < 0$ , $V(N-1) < 0$ then, (2.15) is impossible. Therefore, a cannot be an element of $K_2$ .

Q.E.D.

Theorem 1:

If $N = 2$ then $f^*(1) \leq f^*(2)$ , i.e., the optimal policy is switchover.

Proof:

The proof immediately follows from Lemma 5.

Theorem 2:

If $N = 3$ and $\mu_K \leq \lambda$ , and $\frac{c(\mu_a)}{\mu_a} < A$ for some $a$ , then the optimal policy is switchover, i.e.,

$$f^*(1) \le f^*(2) \le f^*(3) .$$

Proof:

By Lemma 5 $f^*(1) \le f^*(3)$ and $f^*(2) \le f^*(3)$ . We only need to show $f^*(1) \le f^*(2)$ .

Assume $f^*(1) = k$ , then for any $a < k$

$$g(1,k,a) = \frac{c(\mu_k)}{\mu_k} - \frac{c(\mu_a)}{\mu_a} + \left[\frac{\mu_k}{\lambda + \mu_k} e^{-\frac{\alpha}{\mu_k}} - \frac{\mu_a}{\lambda + \mu_a} e^{-\frac{\alpha}{\mu_a}}\right] V_\alpha(0)$$

$$+ \left[\frac{\lambda\mu_k}{(\lambda + \mu_k)^2} e^{-\frac{\alpha}{\mu_k}} - \frac{\lambda\mu_a}{(\lambda + \mu_a)^2} e^{-\frac{\alpha}{\mu_a}}\right] V_\alpha(1)$$

$$+ V(2) \sum_{j=2}^{\infty} \left[\frac{\lambda^j\mu_k}{(\lambda + \mu_k)^{j+1}} e^{-\frac{\alpha}{\mu_k}} - \frac{\lambda^j\mu_a}{(\lambda + \mu_a)^{j+1}} e^{-\frac{\alpha}{\mu_a}}\right] .$$

By assumption $g(1,k,a) < 0$ . We now show that $g(2,k,a) \le g(1,k,a) \le 0$

$$g(2,k,a) = \left[\frac{\mu_k}{\lambda + \mu_k} e^{-\frac{\alpha}{\mu_k}} - \frac{\mu_a}{\lambda + \mu_a} e^{-\frac{\alpha}{\mu_a}}\right] V_\alpha(1)$$

$$+ \left[\frac{\lambda\mu_k}{(\lambda + \mu_k)^2} e^{-\frac{\alpha}{\mu_k}} - \frac{\lambda\mu_a}{(\lambda + \mu_a)^2} e^{-\frac{\alpha}{\mu_a}}\right] V_\alpha(2)$$

$$+ V_\alpha(2) \sum_{j=2}^{\infty} \left[\frac{\lambda^j\mu_k}{(\lambda + \mu_k)^{j+1}} e^{-\frac{\alpha}{\mu_k}} - \frac{\lambda\mu_a}{(\lambda + \mu_a)^{j+1}} e^{-\frac{\alpha}{\mu_a}}\right] .$$

$$g(2,k,a) - g(1,k,a) = \left[\frac{\mu_k}{\lambda + \mu_k} e^{-\frac{\alpha}{\mu_k}} - \frac{\mu_a}{\lambda + \mu_a} e^{-\frac{\alpha}{\mu_a}}\right][V_\alpha(1) - V_\alpha(0)]$$

$$+ \left[\frac{\lambda\mu_k}{(\lambda + \mu_k)^2} e^{-\frac{\alpha}{\mu_k}} - \frac{\lambda\mu_a}{(\lambda + \mu_a)^2} e^{-\frac{\alpha}{\mu_a}}\right][V_\alpha(2) - V_\alpha(1)]$$

since $V(1) < V(0)$ and $V(2) < V(1)$ and

$$\frac{\mu_k}{\lambda + \mu_k} e^{-\frac{\alpha}{\mu_k}} - \frac{\mu_a}{\lambda + \mu_a} e^{-\frac{\alpha}{\mu_a}} > 0$$

$$\frac{\lambda\mu_k}{(\lambda + \mu_k)^2} e^{-\frac{\alpha}{\mu_k}} - \frac{\lambda\mu_a}{(\lambda + \mu_a)^2} e^{-\frac{\alpha}{\mu_a}} \geq 0$$

by assumption. Hence,

$$g(2,k,a) - g(1,k,a) \leq 0 .$$

Therefore, if action $k$ is better than action $a$ in state $1$ (for $k > a$), then action $k$ is better than action $a$ in state $2$.

Q.E.D.

<u>Theorem 3</u>:

Assume $K = 2$ and $A > \frac{c(\mu_a)}{\mu_a}$ for at least one $a$, and $V_\alpha(i + 1) - V_\alpha(i)$ is nondecreasing in $i$, then switchover policy is optimal. That is, there exists a state $J$, $0 \leq J \leq N$, such that

$$f^*(i) = 1 \qquad \text{for } i \leq J$$

$$f^*(i) = 2 \qquad \text{for } i > J .$$

<u>Proof</u>:

Assume $i$ is a state such that $f^*(i) = 2$, then

$$\frac{c(\mu_2)}{\mu_2} - A + \sum_{j=0}^{N-i-1} \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} V_\alpha(i+j-1) +$$

$$+ \sum_{j=N-i}^{\infty} \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} V_\alpha(N-1) < \frac{c(\mu_1)}{\mu_1} - A +$$

$$\sum_{j=0}^{N-i-1} \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}} V_\alpha(i+j-1) + \sum_{j=N-i}^{\infty} \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}} V_\alpha(N-1)$$

or

$$g(i,2,1) = \frac{c(\mu_2)}{\mu_2} - \frac{c(\mu_1)}{\mu_1} + \sum_{j=0}^{N-i-1} \left[ \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} - \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}} \right] \times$$

$$V_\alpha(i+j-1) + \sum_{j=N-i}^{\infty} \left[ \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} - \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}} \right] \times$$

$$V_\alpha(N-1) < 0 \ .$$

We now show that $g(i+1,2,1) < 0$ which implies $f^*(i+1) = 2$ .

$$g(i+1,2,1) = \frac{c(\mu_2)}{\mu_2} - \frac{c(\mu_1)}{\mu_1} + \sum_{j=0}^{N-i-2} \left[ \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} - \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}} \right] \times$$

$$V_\alpha(i+j) + \sum_{j=N-i-1}^{\infty} \left[ \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} - \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}} \right] \times$$

$$V_\alpha(N-1) \ .$$

$$g(i+1,2,1) - g(i,2,1) = \sum_{j=0}^{N-i-1} \left[ \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} - \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}} \right] \times \tag{2.16}$$

$$[V_\alpha(i+j) - V_\alpha(i+j-1)] \ .$$

Now, since

$$h(j) = \frac{\lambda^j \mu_2}{(\lambda+\mu_2)^{j+1}} e^{-\frac{\alpha}{\mu_2}} - \frac{\lambda^j \mu_1}{(\lambda+\mu_1)^{j+1}} e^{-\frac{\alpha}{\mu_1}}$$

changes sign from positive to negative, $\sum_{j=0}^{\infty} h(j) > 0$ , and $V_\alpha(i+j) - V_\alpha(i+j-1) < 0$ then the negative terms in the right side of (2.16) dominates the positive terms. Hence

$$g(i+1,2,1) - g(i,2,1) \leq 0$$

$$g(i+1,2,1) \leq g(i,2,1) < 0$$

or $f^*(i+1) = 2$ .

Q.E.D.

# CHAPTER III

# AVERAGE COST MODEL, TWO SERVICE RATES

## 3.1 Introduction

In this chapter we will consider the same queueing system with the same cost structure, but the criterion for decision rules will be to minimize the long run expected average cost. We assume that only two service rates $\mu_1 < \mu_2$ are available and the service rate can be switched either at an arrival or departure epoch during the process. When service rate $\mu_a$ is in effect, the service time is a random variable which is exponentially distributed with mean $1/\mu_a$. We will use methods of Ross as stated in the last chapter to find the form of optimal policy.

In Section 2 the existence of stationary policy will be proved, and in Section 3 the form of the optimal stationary policy will be found. An efficient computational method will be given in Section 4. In Section 5 the problem of "hesitating customers," who join the queue with certain probability, is analyzed and simple optimal policy is found.

## 3.2 Existence of Stationary Policy

We use the method of semi-Markov decision processes to analyze the queueing problem. The state of the system, as in Chapter II, will be the number of customers in the system and the action space is $K = \{1,2\}$. The transition probabilities are the same as those given in (2.2).

As was stated in Chapter I, under the following three conditions there exists an optimal stationary policy.

(i) $\bar{c}(i,a)$, the expected cost during a transition interval, is bounded.

(ii) $|\bar{V}_\alpha(i) - \bar{V}_\alpha(0)| < M$ for all $i$ and $\alpha$ and for some $M < \infty$. $\bar{V}_\alpha(i)$ are as defined in Chapter II.

(iii) $E_f[T/X_1 = i] < \infty$ , where $T$ is the time of the first return to state $i$ , $X_1$ is the initial state of the system, and $f$ is a stationary policy.

The optimal stationary policy can be found from the optimality conditions stated in Chapter I, and will be discussed later in this section. Under conditions (i), (ii), and (iii) this optimal policy will minimize $\phi^1(i)$ , the expected long run average cost starting from state $i$ , as defined in Chapter I.

Lemma 1:

Conditions (i), (ii) and (iii) hold for the queueing problem.

Proof:

The proof of (i) is immediate. (ii) was proved in Chapter II (Lemma 4). Since the chain, defined in part b of Lemma 4, Chapter II, is irreducible and positive recurrent then the number of transitions before the first return to $i$ is bounded for all $i$ , and so is the expected time of each transition. Hence (iii) holds.

Q.E.D.

## 3.3 Derivation of Optimal Policy

In this section we first modify the optimality conditions for the queueing system, and then the form of optimal policy will be derived.

The optimal stationary policy is one which prescribes the minimizing action in the following equations.

$$h(i) = \min_a \left\{ \bar{c}(i,a) + \sum_j P_{ij}(a)h(j) - g\bar{\tau}(i,a) \right\} .$$

$P_{ij}(a)$ were defined in (2.2) and

$$\bar{c}(0,a) = 0$$

$$\bar{c}(i,a) = \frac{\mu_a}{\lambda + \mu_a}\left\{\frac{c(\mu_a)}{\mu_a} - A\right\} \qquad i = 1, \ldots, N$$

$$\bar{\tau}(0,a) = \frac{1}{\lambda}$$

$$\bar{\tau}(i,a) = \frac{1}{\lambda + \mu_a} \qquad i = 1, \ldots, N .$$

Hence,

$$(3.1)\quad \begin{cases} h(0) = h(1) - \dfrac{g}{\lambda} \\ h(i) = \min\limits_a \left\{\dfrac{c(\mu_a) - \mu_a A}{\lambda + \mu_a} + \dfrac{\lambda}{\lambda + \mu_a} h(i+1) + \dfrac{\mu_a}{\lambda + \mu_a} h(i-1) - \dfrac{g}{\lambda + \mu_a}\right\} \\ \qquad i = 1, \ldots, N-1 \\ h(N) = \min\limits_a \left\{\dfrac{c(\mu_a) - \mu_a A}{\lambda + \mu_a} + \dfrac{\mu_a}{\lambda + \mu_a} h(N-1) + \dfrac{\lambda}{\lambda + \mu_a} h(N) - \dfrac{g}{\lambda + \mu_a}\right\} . \end{cases}$$

(3.1) can be written as follows:

$$\begin{cases} g = \lambda[h(1) - h(0)] \\ \min\limits_a \left\{\dfrac{c(\mu_a) - \mu_a A}{\lambda + \mu_a} + \dfrac{\lambda}{\lambda + \mu_a}[h(i+1) - h(i)] - \dfrac{\mu_a}{\lambda + \mu_a}[h(i) - h(i-1)] \right. \\ \qquad \left. - \dfrac{g}{\lambda + \mu_a}\right\} = 0 \qquad i = 1, \ldots, N-1 \\ \min\limits_a \left\{\dfrac{c(\mu_a) - \mu_a A}{\lambda + \mu_a} - \dfrac{\mu_a}{\lambda + \mu_a}[h(N) - h(N-1)] - \dfrac{g}{\lambda + \mu_a}\right\} = 0 \end{cases}$$

or

$$\begin{cases} g = \lambda[h(1) - h(0)] \\ \dfrac{1}{\lambda + \mu_a} \min_a \{c(\mu_a) - \mu_a A + \lambda[h(i+1) - h(i)] - \mu_a[h(i) - h(i-1)] - g\} = 0 \\ \qquad\qquad i = 1, \ldots, N-1 \\ \dfrac{1}{\lambda + \mu_a} \min_a \{c(\mu_a) - \mu_a A - \mu_a[h(N) - h(N-1)] - g\} = 0 . \end{cases}$$

Finally, since $\dfrac{1}{\lambda + \mu_a} > 0$ for all $a$, (3.1) is equivalent to:

$$(3.2) \quad \begin{cases} g = \lambda[h(1) - h(0)] \\ g = \min_a \{c(\mu_a) - \mu_a A + \lambda[h(i+1) - h(i)] - \mu_a[h(i) - h(i-1)]\} \\ \qquad\qquad i = 1, \ldots, N-1 \\ g = \min_a \{c(\mu_a) - \mu_a A - \mu_a(h(N) - h(N-1))\} . \end{cases}$$

<u>Derivations</u>:

The following lemma is proved using the stationary probabilities for the M/M/1 queueing system without any use of semi-Markov decision process. The lemma is true for any $K \geq 2$ and gives the form of optimal policy in two special cases.

<u>Lemma 2</u>:

(a) If $A \geq \dfrac{c(\mu_K)}{\mu_K}$ and $\dfrac{c(\mu_K)}{\mu_K} = \min_{a \in K} \left\{\dfrac{c(\mu_a)}{\mu_a}\right\}$, then the optimal policy is to use $\mu_K$ in all states, where $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_K$.

(b) If $A \leq \dfrac{c(\mu_1)}{\mu_1}$ and $\dfrac{c(\mu_1)}{\mu_1} = \min_a \left\{\dfrac{c(\mu_a)}{\mu_a}\right\}$, then the optimal policy is to use $\mu_1$ in all states.

<u>Proof</u>:

Let $\{\beta_i\}_{i=0}^{\infty}$ be the stationary probabilities of the queueing system, where $\beta_i$ is the proportion of the time that the system spends in state $i$.

Let $(r_1, r_2, \ldots, r_N)$ be the set of service rates used in states $(1,2, \ldots, N)$ where $r_i \in \mu = [\mu_1, \ldots, \mu_K]$ then

$$\beta_1 = \frac{\lambda}{r_1} \beta_0$$

$$\beta_n = \frac{\lambda^n}{r_1 r_2 \cdots\cdots r_n} \beta_0 \qquad n = 1,2, \ldots, N$$

$$\beta_n = 0 \qquad n > N$$

$$\beta_0 = \frac{1}{1 + \frac{\lambda}{r_1} + \frac{\lambda^2}{r_1 r_2} + \ldots + \frac{\lambda^N}{r_1 r_2 \cdots\cdots r_N}} .$$

We first show that $\beta_N$ , the proportion of the time that system spends in state $N$ , is decreasing in each $r_i$ .

$$\beta_N = \frac{\dfrac{\lambda^N}{r_1 r_2 \cdots\cdots r_N}}{1 + \frac{\lambda}{r_1} + \ldots + \frac{\lambda^N}{r_1 r_2 \cdots\cdots r_N}}$$

Treating $r_i$'s as continuous variables we have

$$\frac{\partial \beta_N}{\partial r_i} = \frac{\dfrac{-\lambda^N}{r_1 \cdots\cdots r_{i-1} r_i^2 r_{i+1} \cdots\cdots r_N}\left[1 + \frac{\lambda}{r_1} + \ldots + \frac{\lambda^N}{r_1 \cdots\cdots r_N}\right]}{\left[1 + \frac{\lambda}{r_1} + \ldots + \frac{\lambda^N}{r_1 \cdots\cdots r_N}\right]^2} +$$

$$\frac{\dfrac{\lambda^N}{r_1 \cdots\cdots r_{i-1} r_i^2 r_{i+1} \cdots\cdots r_N}\left[\frac{\lambda^i}{r_1 \cdots\cdots r_i} + \ldots + \frac{\lambda^N}{r_1 \cdots\cdots r_N}\right]}{\left[1 + \frac{\lambda}{r_1} + \ldots + \frac{\lambda^N}{r_1 \cdots\cdots r_N}\right]^2} .$$

$$\frac{\partial \beta_N}{\partial r_i} = \frac{\dfrac{-\lambda^N}{r_1 \cdots\cdots r_{i-1} r_i^2 r_{i+1} \cdots\cdots r_N}\left[1 + \dfrac{\lambda}{r_1} + \cdots + \dfrac{\lambda^{i-1}}{r_1 \cdots\cdots r_{i-1}}\right]}{\left[1 + \dfrac{\lambda}{r_1} + \cdots + \dfrac{\lambda^N}{r_1 \cdots\cdots r_N}\right]^2} < 0 \quad .$$

$i = 1,2, \ldots, N$ .

Hence, $\beta_N$ is maximized if $r_i = \mu_1$ for all $i$ , and minimized if $r_i = \mu_K$ for all $i$ .

Let $\lambda_e$ be the effective departure rate or long run average departure rate, then $\lambda_e = \lambda(1 - \beta_N)$ . Since $c(r_i)$ is the cost rate when $r_i$ is used then

$$\sum_{i=1}^{N} \beta_i c(r_i)$$

is the average service cost rate. The average return rate will be:

$$G = \lambda(1 - \beta_N)A - \sum_{i=1}^{N} \beta_i c(r_i) \ .$$

Let $g = \max G = \max\limits_{\substack{r_1,\ldots,r_N \\ r_i \in \mu}} \left\{\lambda(1 - \beta_N)A - \sum\limits_{i=1}^{N} \beta_i c(r_i)\right\}$ .

$$\text{(a)} \quad g = \max_{\substack{r_1,\ldots,r_N \\ r_i \in \mu}} \left\{\lambda(1 - \beta_N)A - \sum_{i=1}^{N} \beta_i r_i \frac{c(r_i)}{r_i}\right\}$$

$$\leq \max_{\substack{r_1,\ldots,r_N \\ r_i \in \mu}} \left\{\lambda(1 - \beta_N)A - \frac{c(\mu_K)}{\mu_K} \sum_{i=1}^{N} \beta_i r_i\right\} .$$

Now since $\sum_{i=1}^{N} \beta_i r_i$ is the long run average service rate or the effective service rate, then

$$\sum_{i=1}^{N} \beta_i r_i = \lambda(1 - \beta_N) .$$

Hence,

$$g \leq \max_{\substack{r_1,\ldots,r_N \\ r_i \varepsilon \mu}} \lambda(1 - \beta_N)\left[A - \frac{c(\mu_K)}{\mu_K}\right] . \tag{3.3}$$

But by assumption $A - \frac{c(\mu_K)}{\mu_K} \geq 0$ then, the right side of (3.3) is maximized if $\beta_N$ is minimized. $\beta_N$ is minimized if $r_i = \mu_K$ for all $i$. Therefore, for this choice of rates the right side of (3.3) is maximum, but for the same choice of rates we will have equality in (3.3).

Q.E.D.

(b) $$g = \max_{\substack{r_1,\ldots,r_N \\ r_i \varepsilon \mu}} \left\{\lambda(1 - \beta_N)A - \sum_{i=1}^{N} \beta_i c(r_i)\right\}$$

$$\leq \max_{\substack{r_1,\ldots,r_N \\ r_i \varepsilon \mu}} \left\{\lambda(1 - \beta_N)A - \frac{c(\mu_1)}{\mu_1} \sum_{i=1}^{N} \beta_i r_i\right\}$$

by assumption or

$$g \leq \max_{\substack{r_1,\ldots,r_N \\ r_i \varepsilon \mu}} \lambda(1 - \beta_N)\left[A - \frac{c(\mu_1)}{\mu_1}\right] . \tag{3.4}$$

Since $A - \frac{c(\mu_1)}{\mu_1} < 0$ then the right side of (3.4) is maximized if $\beta_N$ is maximized, but $\beta_N$ is maximized if $r_i = \mu_1$ for all $i$, and for this choice of rates we have equality in (3.4).

Q.E.D.

Lemma 3:

Let $R = \frac{c(\mu_2) - c(\mu_1)}{\mu_2 - \mu_1} - A$. Then, the optimal policy is to use $\mu_1$ in state $i$ if and only if

$$h(i) - h(i-1) \leq R . \tag{3.5}$$

Proof:

Let $f^*(i)$ be the optimal action when in state $i$, then $f^*(i) = 1$ if and only if

$$\begin{aligned} &c(\mu_1) - \mu_1 A + \lambda[h(i+1) - h(i)] - \mu_1[h(i) - h(i-1)] \leq \\ &c(\mu_2) - \mu_2 A + \lambda[h(i+1) - h(i)] - \mu_2[h(i) - h(i-1)] \end{aligned} \tag{3.6}$$

or

$$h(i) - h(i-1) \leq \frac{c(\mu_2) - c(\mu_1)}{\mu_2 - \mu_1} - A .$$

Q.E.D.

Theorem 1:

If $A \geq \frac{c(\mu_a)}{\mu_a}$ for at least one of the rates, then there exists a state $J$, $0 \leq J \leq N$, such that

$$f^*(i) = 1 \qquad i = 1,2, \ldots, J-1$$

$$f^*(i) = 2 \qquad i = J, J+1, \ldots, N .$$

That is, the optimal policy is switchover.

Proof:

Let $k$ be the smallest state such that $f^*(k) = 2$ and $f^*(k-1) = 1$. We consider two cases:

Case 1:

$k = 1$, then

$$h(0) = h(1) - \frac{g}{\lambda} \tag{3.7}$$

$$g = \lambda[h(1) - h(0)] > \lambda R \tag{3.8}$$

by Lemma 3. For state 1, since $f^*(1) = 2$

$$\begin{aligned} g &= c(\mu_2) - \mu_2 A + \lambda[h(2) - h(1)] - \mu_2[h(1) - h(0)] \\ g &< c(\mu_2) - \mu_2 A + \lambda[h(2) - h(1)] - \mu_2 R . \end{aligned} \tag{3.9}$$

From (3.8) and (3.9)

$$\lambda R < c(\mu_2) - \mu_2 A + \lambda[h(2) - h(1)] - \mu_2 R$$

or

$$\lambda R + \mu_2 R - c(\mu_2) + \mu_2 A < \lambda[h(2) - h(1)] . \tag{3.10}$$

Now for $\dfrac{c(\mu_1)}{\mu_1} \le \dfrac{c(\mu_2)}{\mu_2}$ we have $\mu_2 R - c(\mu_2) + \mu_2 A \ge 0$ since

$$\begin{aligned} \mu_2 R - c(\mu_2) + \mu_2 A &= \mu_2 \frac{c(\mu_2) - c(\mu_1)}{\mu_2 - \mu_1} - \mu_2 A - c(\mu_2) + \mu_2 A \\ &= \left(\frac{\mu_2}{\mu_2 - \mu_1} - 1\right) c(\mu_2) - \frac{\mu_2}{\mu_2 - \mu_1} c(\mu_1) \\ &= \frac{1}{\mu_2 - \mu_1}[\mu_1 c(\mu_2) - \mu_2 c(\mu_1)] \ge 0 . \end{aligned}$$

Hence, from (3.10)

$$h(2) - h(1) > R$$

or by Lemma 3, $f^*(2) = 2$. Continuing in this manner we can show $f^*(i) = 2$ for $i = 3,4, \ldots, N$. But for $\frac{c(\mu_2)}{\mu_2} < \frac{c(\mu_1)}{\mu_1}$ by Lemma 2(a) we know $\mu_2$ is optimal in all states. This completes the proof for Case 1.

<u>Case 2</u>:

$k \geq 2$. By Lemma 3

$$(3.11) \qquad \begin{cases} h(k-1) - h(k-2) \leq R & f^*(k-1) = 1 \\ h(k) - h(k-1) > R & f^*(k) = 2 \end{cases}$$

and for state $k$ and $k - 1$,

$$\begin{cases} g = c(\mu_1) - \mu_1 A + \lambda[h(k) - h(k-1)] - \mu_1[h(k-1) - h(k-2)] \\ g = c(\mu_2) - \mu_2 A + \lambda[h(k+1) - h(k)] - \mu_2[h(k) - h(k-1)] \end{cases}$$

or by (3.11)

$$\begin{cases} g > c(\mu_1) - \mu_1 A + \lambda R - \mu_1 R \\ g < c(\mu_2) - \mu_2 A - \mu_2 R + \lambda[h(k+1) - h(k)] \,. \end{cases}$$

Then,

$$c(\mu_1) - \mu_1 A + \lambda R - \mu_1 R < c(\mu_2) - \mu_2 A - \mu_2 R + \lambda[h(k+1) - h(k)]$$

or

$$\lambda[h(k+1) - h(k)] < c(\mu_1) - c(\mu_2) - (\mu_1 - \mu_2)A + (\mu_2 - \mu_1)R + \lambda R \,.$$

But

$$(\mu_2 - \mu_1)R - c \text{ [illegible] } c(\mu_1) \text{ [illegible] } (\mu_2 - \mu_1)R \ .$$

Hence,

$$h(k+1) - h(k) > R$$

or $f^*(k+1) = 2$ . Continuing in the same manner we can show $f^*(i) = 2$ for $i = k+2, \ldots, N$ which completes the proof of the theorem.

<u>Theorem 2</u>:

If $A < \dfrac{c(\mu_a)}{\mu_a}$ for $a = 1,2$ . Then, there exists a state $J$ , $0 \le J \le N$ , such that

$$f^*(i) = 2 \qquad \text{for } i = 1, \ldots, J-1$$

$$f^*(i) = 1 \qquad \text{for } i = J, J+1, \ldots, N \ .$$

That is, the optimal policy is a reverse switchover.

<u>Proof</u>:

The proof is similar to the proof of Theorem 1. Let $k$ be the smallest state such that $f^*(k) = 1$ and $f^*(k-1) = 2$ , consider two cases:

<u>Case 1</u>:

$k = 1$ , $f^*(1) = 1$ and

$$h(1) - h(0) \le R \ .$$

For state $0$ ,

(3.12) $$g = \lambda[h(1) - h(0)] \le \lambda R \ .$$

For state $1$ ,

$$(3.13)\quad \begin{aligned} g &= c(\mu_1) - \mu_1 A + \lambda[h(2) - h(1)] - \mu_1[h(1) - h(0)] \\ g &\geq c(\mu_1) - \mu_1 A + \lambda[h(2) - h(1)] - \mu_1 R \ . \end{aligned}$$

From (3.12) and (3.13)

$$(3.14)\quad \begin{aligned} & c(\mu_1) - \mu_1 A + \lambda[h(2) - h(1)] - \mu_1 R \leq \lambda R \\ & \lambda[h(2) - h(1)] \leq \lambda R + \mu_1 R - c(\mu_1) + \mu_1 A \ . \end{aligned}$$

But

$$\begin{aligned} \mu_1 R - c(\mu_1) + \mu_1 A &= \mu_1 \frac{c(\mu_2) - c(\mu_1)}{\mu_2 - \mu_1} - \mu_1 A - c(\mu_1) + \mu_1 A \\ &= \frac{1}{\mu_2 - \mu_1}[\mu_1 c(\mu_2) - \mu_2 c(\mu_1)] \leq 0 \qquad \text{if } \frac{c(\mu_1)}{\mu_1} \geq \frac{c(\mu_2)}{\mu_2} \ . \end{aligned}$$

Then (3.14) reduces to

$$h(2) - h(1) \leq R$$

or $f^*(2) = 1$ . Continuing in the same manner we can show $f^*(i) = 1$ for $i = 3,4, \ldots, N$ . If $\frac{c(\mu_1)}{\mu_1} < \frac{c(\mu_2)}{\mu_2}$ then by Lemma 2(b) the optimal policy is to use $\mu_1$ in all states, which completes the proof of Case 1.

Case 2:

The proof is similar to the proof of Case 2, Theorem 1 and details are omitted here.

### 3.4 A Computational Approach

In the last section we proved that the optimal policy has a simple form. It uses (in the case $A \geq \frac{c(\mu_a)}{\mu_a}$ for some a) $\mu_1$ up to a state $J - 1$ and then it is switched to rate $\mu_2$ . We give a simple computation method to find $J$ .

Let $P_1$ be the proportion of time the system spends using rate $\mu_1$ in a given switchover policy. Since the cost rate is $c(\mu_1) - \mu_1 A$ and $c(\mu_2) - \mu_2 A$ for rates $\mu_1$ and $\mu_2$ respectively then

$$(3.15) \qquad g = P_1[c(\mu_1) - \mu_1 A] + (1 - P_1)[c(\mu_2) - \mu_2 A]$$

where $g$ is the average rate of return as defined before. Now let $P_1^j$ be the proportion of time that the system spends using $\mu_1$ where state $j$ is the switching state, then

$$P_1^j = \sum_{i=1}^{j-1} \beta_i^j$$

where $\beta_i^j$ are the stationary probability for this given policy

$$\beta_i^j = \frac{\lambda^i}{\mu_1^i} \beta_0^j \qquad i \le j - 1$$

$$\beta_i^j = \frac{\lambda^{j-1}}{\mu_1^{j-1} \mu_2^{i-j+1}} \beta_0^j \qquad i \ge j \qquad i = 1, \ldots, N$$

$$\beta_i^j = 0 \qquad i > N$$

$$\beta_0^j = \frac{1}{1 + \frac{\lambda}{\mu_1} + \ldots + \frac{\lambda^{j-1}}{\mu_1^{j-1}} + \frac{\lambda^j}{\mu_1^{j-1}\mu_2} + \ldots + \frac{\lambda^N}{\mu_1^{j-1}\mu_2^{N-j+1}}} .$$

We first compute

$$P_1^1, P_1^2, \ldots, P_1^N .$$

Now if

$$c(\mu_1) - \mu_1 A \le c(\mu_2) - \mu_2 A$$

then in (3.13) we take $P_1$ as large as possible, and $J$ is found from

$$P_1^J = \max_j \left\{P_1^j\right\} .$$

And if

$$c(\mu_1) - \mu_1 A > c(\mu_2) - \mu_2 A$$

then the optimal $J$ is found from

$$P_1^J = \min_j \left\{P_1^j\right\} .$$

The same method can be used in the case $A < \frac{c(\mu_a)}{\mu_a}$ , $a = 1,2$ , except the switching is from rate $\mu_2$ to rate $\mu_1$ .

### 3.5 The Hesitating Customers Problem

In this section we assume that the arrival rate depends on the state of the system. It is assumed the arrival rate is $\lambda P_i$ when the system is in state $i$ , where $0 \leq P_i \leq 1$ . In other words an arriving customer will join the queue with a probability $P_i$ whenever the system is in state $i$ . The same cost structure is considered and the service facility will be the same as in the last sections. Let $H(i)$ be the relative cost value corresponding to state $i$ , equivalent to $h(i)$ in the last sections, then the optimal stationary policy could be found from the following set of equations:

$$(3.16) \quad \begin{cases} g = \lambda P_0[H(1) - H(0)] \\ g = \min_a \{c(\mu_a) - \mu_a A + \lambda P_i[H(i + 1) - H(i)] - \mu_a[H(i) - H(i - 1)]\} \\ \qquad i = 1, \ldots, N-1 \\ g = \min_a \{c(\mu_a) - \mu_a A - \mu_a[H(N) - H(N - 1)]\} . \end{cases}$$

With this definition of relative cost functions one can state the equivalent of Lemma 3 as follows:

Lemma 3':

Let $R = \dfrac{c(\mu_2) - c(\mu_1)}{\mu_2 - \mu_1} - A$ then $f^*(i) = 1$ if and only if

$$H(i) - H(i-1) \leq R .$$

The proof is immediate.

Lemma 4:

If $A \geq \dfrac{c(\mu_a)}{\mu_a}$ for some $a$ and $H(0) = 0$ , then $H(i)$ is nonincreasing as a function of $i$ .

Proof:

$$g = \lambda P_0[H(1) - H(0)] = \lambda P_0 H(1) . \tag{3.17}$$

But $A \geq \dfrac{c(\mu_a)}{\mu_a}$ then, $g \leq 0$ and $H(1) \leq 0$ . Now it can easily be shown that $\bar{V}_\alpha(i)$ as defined in Chapter II is nonincreasing for the hesitating customers problem and since $H(i)$ has the same structural form as $\bar{V}_\alpha(i)$ (see Ross [13]) then $H(i)$ will be nonincreasing.

Q.E.D.

Theorem 3:

If the $P_i$'s are increasing in $i$ and $\dfrac{c(\mu_a)}{\mu_a} < A$ for at least one $a$ and $\dfrac{c(\mu_1)}{\mu_1} \leq \dfrac{c(\mu_2)}{\mu_2}$ , then the switchover policy is optimal for the hesitating customers problem.

Proof:

Let $k$ be the smallest state such that $f^*(k) = 2$ and $f^*(k - 1) = 1$, we prove the theorem in two cases as we did in the proof of Theorem 1.

Case 1:

$k = 1$.

$$g = \lambda P_0[H(1) - h(0)] > \lambda P_0 R \tag{3.18}$$

by Lemma 3'. Note that since $g < 0$ then $R < 0$. For state 1

$$\begin{aligned} g &= c(\mu_2) - \mu_2 A + \lambda P_1[H(2) - H(1)] - \mu_2[H(1) - H(0)] \\ &< c(\mu_2) - \mu_2 A + \lambda P_1[H(2) - H(1)] - \mu_2 R . \end{aligned} \tag{3.19}$$

From (3.18) and (3.19) we have

$$c(\mu_2) - \mu_2 A + \lambda P_1[H(2) - H(1)] - \mu_2 R > \lambda P_0 R$$

or

$$\lambda P_1[H(2) - H(1)] > \lambda P_0 R + \mu_2 R - c(\mu_2) + \mu_2 A . \tag{3.20}$$

But by assumption

$$\begin{aligned} \mu_2 R - c(\mu_2) + \mu_2 A &= \frac{\mu_2}{\mu_2 - \mu_1}[c(\mu_2) - c(\mu_1)] - \mu_2 A - c(\mu_2) + \mu_2 A \\ &= \frac{1}{\mu_2 - \mu_1}[\mu_1 c(\mu_2) - \mu_2 c(\mu_1)] \geq 0 . \end{aligned}$$

Hence, (3.20) reduces to

$$H(2) - H(1) > \frac{P_0}{P_1} R \geq R .$$

Since $R < 0$ and $\frac{P_0}{P_1} \le 1$ , or $f^*(2) = 2$ . Continuing in the same way it can be shown $f^*(i) = 2$ for all $i \ge 2$ .

Case 2:

$k \ge 2$ , then

$$\begin{aligned} H(k-1) - H(k-2) &\le R \\ H(k) - H(k) &> R \end{aligned} \tag{3.21}$$

and $R \le 0$ .

For state $k - 1$ and $k$ we have

$$g = c(\mu_1) - \mu_1 A + P_{k-1}\lambda[H(k) - H(k-1)] - \mu_1[H(k-1) - H(k-2)]$$

$$g = c(\mu_2) - \mu_2 A + P_k\lambda[H(k+1) - H(k)] - \mu_2[H(k) - H(k-1)]$$

or by (3.21)

$$g > c(\mu_1) - \mu_1 A + P_{k-1}\lambda R - \mu_1 R$$

$$g < c(\mu_2) - \mu_2 A + \lambda P_k[H(k+1) - H(k)] - \mu_2 R$$

or

$$c(\mu_2) - \mu_2 A + \lambda P_k[H(k+1) - H(k)] - \mu_2 R > c(\mu_1) - \mu_1 A + \lambda P_{k-1} R - \mu_1 R$$

or

$$\lambda P_k[H(k+1) - H(k)] > \lambda P_{k-1} R$$

$$H(k+1) - H(k) > \frac{P_{k-1}}{P_k} R \ge R \text{ .}$$

Since $R \le 0$ and $\frac{P_{k-1}}{P_k} \le 1$ . Therefore, $f^*(k+1) = 2$ . Continuing in the same manner it can be shown $f^*(i) = 2$ for all $i \ge k + 1$ .

Q.E.D.

Now, let us assume that $A < \frac{c(\mu_a)}{\mu_a}$ for $a = 1,2$ . Then Lemma 3' still hold. Equivalent of Lemma 4 can be stated as follows and the proof is similar to the proof of Lemma 4.

<u>Lemma 4'</u>:

If $A < \frac{c(\mu_a)}{\mu_a}$ for $a = 1,2$ , then $H(i)$ is nondecreasing.

<u>Theorem 4</u>:

If the $P_i$'s are increasing in $i$ , $A < \frac{c(\mu_a)}{\mu_a}$ for $a = 1,2$ , and $\frac{c(\mu_2)}{\mu_2} \leq \frac{c(\mu_1)}{\mu_1}$ , then the optimal policy for the hesitating customers problem is reverse switchover. Proof of this theorem is similar to the proof of Theorems 2 and 3. Details are omitted here.

# CHAPTER IV

## AVERAGE COST MODEL, K SERVICE RATES

### 4.1 Introduction

In this chapter the results of Chapter III will be generalized for the $K$ service rates, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_K$. The cost structure is the same, i.e., $c(\mu_k)$ is the cost rate of using service rate $\mu_k$. It is assumed that $c(\cdot)$ is a nondecreasing function. A reward of $A$ is earned for each customer served. Again the method of semi-Markov decision processes is employed in order to minimize the long run expected average cost. The state of the system is the number of customers in the system and the action space is $\mathcal{K} = \{1,2, \ldots, K\}$. The arrival stream as before is the Poisson process with rate $\lambda$ and the service time is exponentially distributed with expected service time $1/\mu_k$ whenever rate $\mu_k$ is in effect. The results of Section 2, Chapter III for the existence of optimal stationary policy are true for $K$ rates case and will not be repeated here. The modifications of optimality conditions which were made in the two rates case will still hold and will be stated below.

If $\pi^*$ is the policy which prescribes the minimizing actions in the following relations, then this policy is stationary and minimizes the long run expected average cost.

$$
(4.1) \quad \begin{cases} g = \lambda[h(1) - h(0)] \\ g = \min\limits_{a\in\mathcal{K}} \{c(\mu_a) - \mu_a A + \lambda[h(i+1) - h(i)] - \mu_a[h(i) - h(i-1)]\} \\ \qquad\qquad i = 1,2, \ldots, N-1 \\ g = \min\limits_{a\in\mathcal{K}} \{c(\mu_a) - \mu_a A - \mu_a[h(N) - h(N-1)]\} \end{cases}
$$

In the next section we will first eliminate some of the service rate from further considerations because they will not appear in the optimal policy, and then the form of optimal policy will be derived.

## 4.2 Elimination of Nonoptimal Rates and Derivations of Optimal Policy

In Chapter III, Lemma 2, we showed in certain cases some service rates will not be used in the optimal policy. For example if the function $c(\cdot)$ is concave and $A > \frac{c(\mu_a)}{\mu_a}$ for some $a$, then the optimal policy is to use rate $\mu_K$, the fastest rate, in all states. Lemma 2 is in fact more general than this. If the service rates are such that the points $(\mu_k, c(\mu_k))$ all lie above the line connecting origin and point $(\mu_K, c(\mu_K))$ then, only $\mu_K$ will be used in the optimal policy. We now in a sequence of five lemmas eliminate some nonoptimal rates.

Let $C$ be the set of all points $(\mu_k, c(\mu_k))$, and consider the piecewise linear convex function $H$ joining $(\mu_1, c(\mu_1))$ and $(\mu_K, c(\mu_K))$ which bounds the set $C$ from below and changes slope only at the points of the set (see Figure 1).

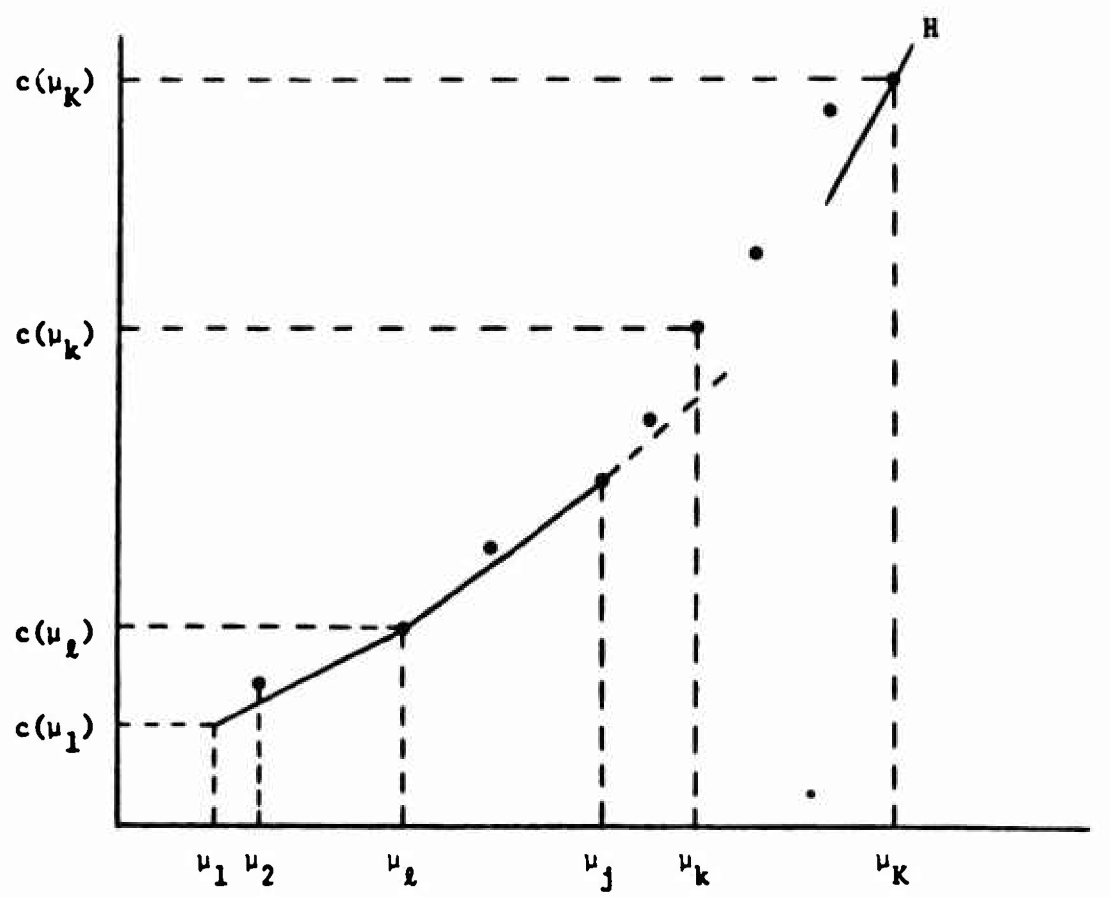


FIGURE 1: SET $C$ OF POINTS $(\mu_j, c(\mu_j))$ AND FUNCTION $H$

Lemma 1:

The set of service rates corresponding to the points $(\mu_k, c(\mu_k))$ not on the function H (above H) will not be used in the optimal policy and can be eliminated.

Proof:

$f^*(i) = k$ , i.e., rate $\mu_k$ is optimal in state $i$ if,

$$
\begin{aligned}
(4.2)\qquad & c(\mu_k) - \mu_k A + \lambda[h(i+1) - h(i)] - \mu_k[h(i) - h(i-1)] \leq \\
& c(\mu_j) - \mu_j A + \lambda[h(i+1) - h(i)] - \mu_j[h(i) - h(i-1)] \qquad \text{for all } j \neq k
\end{aligned}
$$

or

$$
c(\mu_k) - \mu_k A - c(\mu_j) + \mu_j A \leq (\mu_k - \mu_j)[h(i) - h(i-1)] \qquad \text{for } j \neq k
$$

or

$$
(4.3)\qquad \begin{cases} h(i) - h(i-1) > \dfrac{c(\mu_k) - c(\mu_j)}{\mu_k - \mu_j} - A & \text{if } \mu_k > \mu_j \\[2ex] h(i) - h(i-1) \leq \dfrac{c(\mu_j) - c(\mu_k)}{\mu_j - \mu_k} - A & \text{if } \mu_j > \mu_k \,. \end{cases}
$$

(4.3) can be stated as follows

$$
\begin{aligned}
(4.4)\qquad & \max_{0=1,2,\ldots,k-1} \left\{ \frac{c(\mu_k) - c(\mu_j)}{\mu_k - \mu_j} - A \right\} < h(i) - h(i-1) \leq \\
& \min_{j=k+1,\ldots,K} \left\{ \frac{c(\mu_j) - c(\mu_k)}{\mu_j - \mu_k} - A \right\} .
\end{aligned}
$$

Now assume that $(\mu_k, c(\mu_k))$ is above H , and let $\mu_m$ be the largest rate to the left of $\mu_k$ where $(\mu_m, c(\mu_m))$ is on H and $\mu_n$ be the smallest rate to the right of $\mu_k$ and $(\mu_n, c(\mu_n))$ on H (see Figure 2).

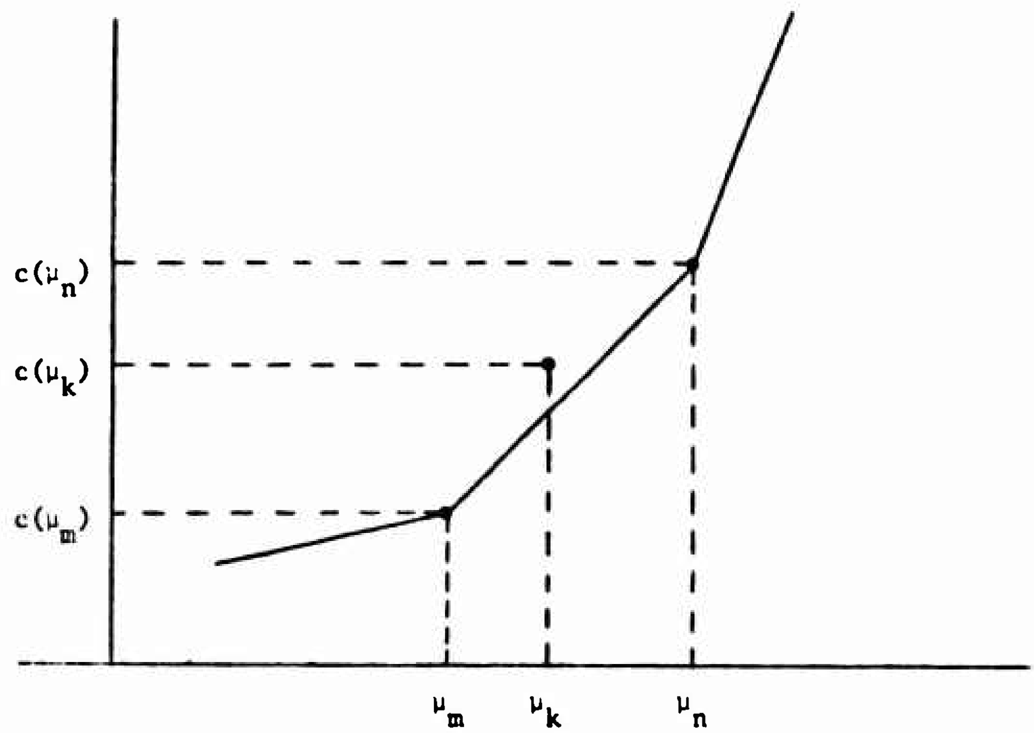


FIGURE 2

Then from definition of H ,

$$\frac{c(\mu_k) - c(\mu_m)}{\mu_k - \mu_m} > \frac{c(\mu_n) - c(\mu_k)}{\mu_n - \mu_k}$$

or

$$\frac{c(\mu_k) - c(\mu_m)}{\mu_k - \mu_m} - A > \frac{c(\mu_n) - c(\mu_k)}{\mu_n - \mu_k} - A .$$

But this violates (4.4). Therefore, $f^*(i) = k$ is not possible and $\mu_k$ will not be optimal in any state and can be eliminated. Now let $\mu_1 < \mu_2 < \ldots < \mu_L$ be those rates such that the corresponding $(\mu, c(\mu))$ points are on H . (Rearrange the indexes so that this is true.) Then,

$$\text{(4.5)} \quad \frac{c(\mu_2) - c(\mu_1)}{\mu_2 - \mu_1} - A < \frac{c(\mu_3) - c(\mu_2)}{\mu_3 - \mu_2} - A < \ldots < \frac{c(\mu_L) - c(\mu_{L-1})}{\mu_L - \mu_{L-1}} - A .$$

We define

$$(4.6) \qquad R_j = \frac{c(\mu_j) - c(\mu_{j-1})}{\mu_j - \mu_{j-1}} - A .$$

Lemma 2:

$f^*(i) = j$ if and only if

$$(4.7) \qquad \frac{c(\mu_j) - c(\mu_{j-1})}{\mu_j - \mu_{j-1}} - A < h(i) - h(i-1) \le \frac{c(\mu_{j+1}) - c(\mu_j)}{\mu_{j+1} - \mu_j} - A$$

or

$$R_j < h(i) - h(i-1) \le R_{j+1} .$$

The proof of this lemma is immediate from (4. , .d (4.5).

The following two lemmas are needed to prove Lemma 5 and Theorem 1.

Lemma 3:

Let $(r_1, \ldots, r_N)$ be the optimal rates in states $(1,2, \ldots, N)$ for a given cost function $c(\mu_1) \le \ldots \le c(\mu_L)$, and assume rate $\mu_t$ is not used in the optimal policy. Now let us consider a new problem with the same structure but a different cost function $c'(\cdot)$, where

$$\begin{cases} c'(\mu_i) = c(\mu_i) & i \ne t \\ c'(\mu_t) > c(\mu_t) . \end{cases}$$

Then $\mu_t$ is not used in the optimal policy for the new problem.

Proof:

The lemma is intuitively clear since increasing the cost of a nonoptimal rate should not change the optimal policy. Formally, let $\beta_i$ be the stationary probability, the proportion of time the system spends in state $i$, as defined in Chapter III, then

$$g = \sum_{i=1}^{N} \beta_i c(r_i) - \lambda(1 - \beta_N)A$$

where $g$ is the minimal average cost rate, and $(r_1, r_2, \ldots, r_N)$ are the set of optimal service rates. Now assume that a different set of rates $(r_1', r_2', \ldots, r_N')$ are optimal for the new problem and $\beta_i'$'s are the corresponding stationary probabilities. Then if $\mu_t$ is used in the optimal policy in state $j$, $g'$ the optimal cost rate for the new problem is

$$g' = \sum_{i=1}^{N} \beta_i' c(r_i') - \lambda(1 - \beta_N')A < g .$$

($g$ is an upper bound since $(r_1, r_2, \ldots, r_N)$ can be used for the new problem which gives the cost rate equal to $g$.) Then

$$\begin{aligned} g' &= \sum_{i \neq j} \beta_i' c(r_i') + \beta_j' c'(\mu_t) - \lambda(1 - \beta_N')A \\ &> \sum_{i \neq j} \beta_i' c(r_i') + \beta_j' c(\mu_t) - \lambda(1 - \beta_N')A . \end{aligned} \tag{4.8}$$

But the right side of (4.8) is the cost rate using the original cost function and gives lower cost rate than $g$, which contradicts the fact that $g$ was the optimal cost rate.

Q.E.D.

<u>Lemma 4</u>:

Let $c = \dfrac{c(\mu_k)}{\mu_k} = \min\limits_{a} \left\{ \dfrac{c(\mu_a)}{\mu_a} \right\}$, $a = 1,2, \ldots, L$ and $c \le A$. Then

$$g > \lambda(c - A) . \tag{4.9}$$

Proof:

This is also intuitively clear since the right side of (4.9) is the negative of maximum return when the cheapest rate is used and all arriving customers are served. Let $\beta_i$ be as defined in Lemma 3, then if $(r_1, r_2, \ldots, r_N)$ is the set of optimal rates

$$g = \sum_{i=1}^{N} \beta_i c(r_i) - \lambda(1 - \beta_N)A = \sum_{i=1}^{N} r_i \beta_i \frac{c(r_i)}{r_i} - \lambda(1 - \beta_N)A$$

$$\geq c \sum_{i=1}^{N} r_i \beta_i - \lambda(1 - \beta_N)A = \lambda(1 - \beta_N)(c - A) > \lambda(c - A) .$$

The last inequality follows from the fact that

$$\beta_N > 0 .$$

Q.E.D.

Lemma 5:

Let $c = \dfrac{c(\mu_k)}{\mu_k} = \min\limits_{a} \left\{\dfrac{c(\mu_a)}{\mu_a}\right\}$, $a = 1,2, \ldots, L$, and $c \leq A$ then all rates $\mu_j$ such that $\mu_j < \mu_k$ can be eliminated from further consideration.

Proof:

This lemma is a generalization of Lemma 2, Chapter III.

Let us assume for simplicity that $k = 2$. We must show $\mu_1$ will not be used in the optimal policy.

Consider the same system with the same service rates available but a different cost function as follows:

$$\begin{cases} c'(\mu_1) = c \cdot \mu_1 \quad \text{where} \quad c = \dfrac{c(\mu_2)}{\mu_2}, \; c(\mu_1) > c \cdot \mu_1 \\ c'(\mu_i) = c(\mu_i) \qquad i = 2, \ldots, L \end{cases}$$

(see Figure 3).

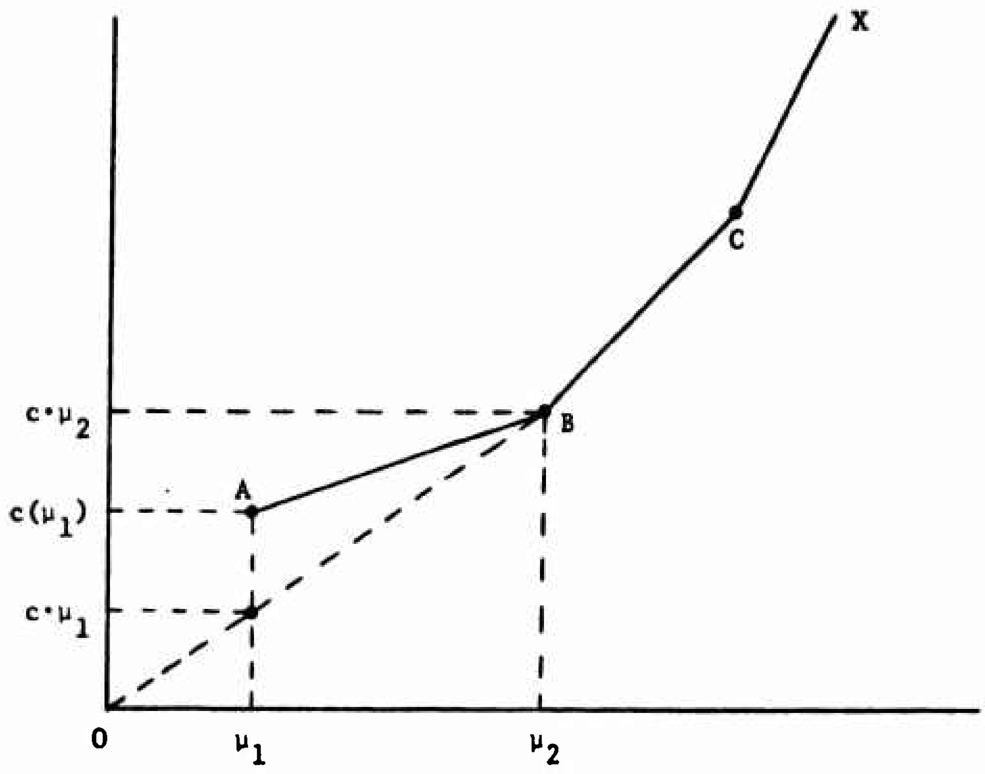


FIGURE 3

Function H : ABCX , the cost function for the original problem, $c(\cdot)$ .

Function H': OBCX , the cost function for the new problem, $c'(\cdot)$ .

We first show that $\mu_1$ will not be used in the optimal policy for the new problem. The proof of the lemma then follows from Lemma 3. In order for $\mu_1$ to be optimal in a state $i$ in the new problem we must have

$$h(i) - h(i-1) \leq \frac{c(\mu_2) - c\mu_1}{\mu_2 - \mu_1} - A = c - A .$$

By Lemma 4

(4.10) $$g = \lambda[h(1) - h(0)] > \lambda(c - A)$$

then

$$h(1) - h(0) = \frac{g}{\lambda} > c - A \ .$$

Therefore, $f^*(1) \neq 1$ . Now consider two cases:

Case 1:

$f^*(1) = 2$ , then

$$g = c(\mu_2) - \mu_2 A + \lambda[h(2) - h(1)] - \mu_2[h(1) - h(0)] \ .$$

by (4.10)

$$g < c(\mu_2) - \mu_2 A + \lambda[h(2) - h(1)] - \mu_2(c - A)$$

and

$$\lambda(c - A) < c(\mu_2) - \mu_2 A + \lambda[h(2) - h(1)] - \mu_2(c - A)$$

or

$$\lambda[h(2) - h(1)] > \lambda(c - A) + \mu_2(c - A) - c\mu_2 + \mu_2 A$$

$$h(2) - h(1) > c - A \ .$$

Therefore, $f^*(2) \geq 2$ .

Now, let $f^*(2) = t$ where $t \geq 2$ . For state 2 we have

$$g = c(\mu_t) - \mu_t A + \lambda[h(3) - h(2)] - \mu_t[h(2) - h(1)]$$

$$< c(\mu_t) - \mu_t A + \lambda[h(3) - h(2)] - \mu_t R_t \ .$$

Then by (4.10)

$$\lambda(c - A) < c(\mu_t) - \mu_t A + \lambda[h(3) - h(2)] - \mu_t R_t$$

or

(4.11) $\quad \lambda[h(3) - h(2)] > \lambda(c - A) + \mu_t R_t - c(\mu_t) + \mu_t A$ .

But

$$\mu_t R_t - c(\mu_t) + \mu_t A = \frac{\mu_t}{\mu_t - \mu_{t-1}}[c(\mu_t) - c(\mu_{t-1})] - c(\mu_t)$$

$$= \frac{1}{\mu_t - \mu_{t-1}}[\mu_{t-1} c(\mu_t) - \mu_t c(\mu_{t-1})] \geq 0$$

since for $t = 2$

$$\mu_1 \cdot c \cdot \mu_2 - \mu_2 \cdot c \cdot \mu_1 = 0$$

and for $t > 2$

$$\frac{c(\mu_t)}{\mu_t} > \frac{c(\mu_{t-1})}{\mu_{t-1}} .$$

($H'$ being convex and passing through origin, see Figure 3.) Hence, (4.11) reduces to

$$h(3) - h(2) > c - A \quad \text{or} \quad f^*(3) \geq 2 .$$

Continuing in the same manner we can show

$$f^*(i) \geq 2 \qquad \text{for} \quad i = 3,4, \ldots, N .$$

Case 2:

$f^*(1) = t$ where $t > 2$ . Then $h(1) - h(0) > R_t > c - A$

$$g = \lambda[h(1) - h(0)] > \lambda R_t .$$

So

For state 1

$$g = c(\mu_t) - \mu_t A + \lambda[h(2) - h(1)] - \mu_t[h(1) - h(0)]$$

and

$$g < c(\mu_t) - \mu_t A + \lambda[h(2) - h(1)] - \mu_t R_t .$$

then

$$\lambda R_t < c(\mu_t) - \mu_t A + \lambda[h(2) - h(1)] - \mu_t R_t$$

or

$$\lambda(h(2) - h(1)) > \lambda R_t + \mu_t R_t - c(\mu_t) + \mu_t A .$$

But

$$\mu_t R_t - c(\mu_t) + \mu_t A \geq 0 \qquad \text{for} \quad t > 2$$

as was shown in Case 1. Hence,

$$h(2) - h(1) > R_t \quad \text{or} \quad f^*(2) \geq t .$$

Continuing in the same way it can be shown $f^*(i) \geq t$ for $i = 3,4, \ldots, N$ . This completes the proof of the lemma for the new problem, but by Lemma 4 the same will be true for the original problem.

Q.E.D.

By this lemma we can now assume that $H$ passes through origin and

$$\frac{c(\mu_1)}{\mu_1} \leq \frac{c(\mu_2)}{\mu_2} \leq \cdots \leq \frac{c(\mu_L)}{\mu_L} .$$

Theorem 1:

Let $\mu_1 < \mu_2 < \dots < \mu_L$ be $L$ service rates such that $R_2 < R_3 < \dots < R_L$ , where $R_j$'s are as defined in (4.6). Then $f^*(i)$ is a nondecreasing function of $i$ , i.e.,

$$f^*(1) \le f^*(2) \le \dots \le f^*(N)$$

which is a generalization of switchover policy.

Proof:

We show that if $f^*(i) = t$ then $f^*(j) \ge t$ for all $j > i$ . We consider two cases:

Case 1:

If $f^*(1) = t$ then by Lemma 5, Cases 1 and 2,

$$f^*(i) \ge t \qquad \text{for all } i > 1 .$$

Case 2:

Now let $i$ be the smallest state such that $f^*(i) > f^*(1)$ , $N \ge i > 1$ . Assume that $f^*(i) = k$ and $f^*(i - 1) = t$ where $t < k$ . For state $i - 1$ and $i$ we have

$$\begin{cases} R_t = \dfrac{c(\mu_t) - c(\mu_{t-1})}{\mu_t - \mu_{t-1}} - A < h(i - 1) - h(i - 2) \le \dfrac{c(\mu_{t+1}) - c(\mu_t)}{\mu_{t+1} - \mu_t} - A = R_{t+1} \\ R_k = \dfrac{c(\mu_k) - c(\mu_{k-1})}{\mu_k - \mu_{k-1}} - A < h(i) - h(i - 1) \le \dfrac{c(\mu_{k+1}) - c(\mu_k)}{\mu_{k+1} - \mu_k} - A = R_{k+1} \end{cases}$$

$$\begin{cases} g = c(\mu_t) - \mu_t A + \lambda[h(i) - h(i - 1)] - \mu_t[h(i - 1) - h(i - 2)] \\ g = c(\mu_k) - \mu_k A + \lambda[h(i + 1) - h(i)] - \mu_k[h(i) - h(i - 1)] . \end{cases}$$

But

$$R_k \geq R_{t+1} > R_t \qquad k \geq t + 1$$

then

$$\begin{cases} g > c(\mu_t) - \mu_t A + \lambda R_k - \mu_t R_k \\ g < c(\mu_k) - \mu_k A + \lambda[h(i + 1) - h(i)] - \mu_k R_k \end{cases}$$

or

$$c(\mu_k) - \mu_k A + \lambda[h(i + 1) - h(i)] - \mu_k R_k > c(\mu_t) - \mu_t A + \lambda R_k - \mu_t R_k$$

(4.12) $\lambda[h(i + 1) - h(i)] > -c(\mu_k) + c(\mu_t) + \mu_k A - \mu_t A + \lambda R_k + (\mu_k - \mu_t)R_k$ .

But

$$(\mu_k - \mu_t)R_k - [c(\mu_k) - c(\mu_t)] + (\mu_k - \mu_t)A = \frac{c(\mu_k) - c(\mu_{k-1})}{\mu_k - \mu_{k-1}} - \frac{c(\mu_k) - c(\mu_t)}{\mu_k - \mu_t} \geq 0$$

the last inequality holds by Lemmas 1 and 5. Then, (4.12) reduces to

$$h(i + 1) - h(i) > R_k$$

or

$$f^*(i + 1) \geq k \ .$$

Continuing in the same manner we can show $f^*(j) \geq k$ for $j = i + 1, i + 2, \ldots, N$ , which completes the proof of Theorem 1.

Now consider that $A < \dfrac{c(\mu_a)}{\mu_a}$ for all $a$ , then Lemma 3 will still hold, and the equivalent of Lemma 4 is Lemma 6.

<u>Lemma 6</u>:

Let $c = \dfrac{c(\mu_k)}{\mu_k} = \min\limits_a \left\{\dfrac{c(\mu_a)}{\mu_k}\right\}$ , $a = 1,2, \ldots, L$ and $c > A$ then $g < \lambda(c - A)$ .

Proof:

Let $\beta_i$ be the stationary probability corresponding to using $\mu_k$ in all states then

$$\beta_i = \frac{\left(\frac{\lambda}{\mu_k}\right)^i}{1 + \frac{\lambda}{\mu_k} + \dots + \left(\frac{\lambda}{\mu_k}\right)^N} \qquad \text{for } i \le N$$

$$\beta_i = 0 \qquad \text{for } i > N$$

and let $G$ be the corresponding average cost rate then

$$g \le G = \lambda(1 - \beta_N)(c - A) < \lambda(c - A)$$

since $\beta_N > 0$ .

Q.E.D.

The following lemma is the equivalent of Lemma 5 for the case $A < \frac{c(\mu_a)}{\mu_a}$ for all $a$ .

Lemma 7:

Let $c = \frac{c(\mu_k)}{\mu_k} = \min_a \left\{\frac{c(\mu_a)}{\mu_a}\right\}$ and $c > A$ then all rates faster than $\mu_k (\mu_j > \mu_k)$ can be eliminated from further consideration.

Proof:

This lemma is intuitively clear since all rates faster than $\mu_k$ are more expensive and their use will increase the rate of number of customers served.

The proof closely follows the method to prove Lemma 5. First we consider a new problem with the cost of rates faster than $\mu_k$ reduced to $c \cdot \mu_j$ and show for this problem the optimal policy does not use rates $\mu_j$ , $\mu_j > \mu_k$ , and then by Lemma 3 complete the proof. Details are omitted here.

Theorem 2:

Let $\mu_1 < \mu_2 < \ldots < \mu_L$ be $L$ service rates such that all nonoptimal rates are eliminated and $R_2 < R_3 < \ldots < R_L$ then $f^*(i)$ is nonincreasing function of $i$ , i.e.,

$$f^*(1) \geq f^*(2) \geq \ldots \geq f^*(N) .$$

Proof:

It will be shown that if $f^*(i) = t$ , then $f^*(j) \leq t$ for all $j > i$ . Details are omitted here since method of proof is closely related to the proof of Theorem 1 of this chapter and Theorem 2 of the last chapter.

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